The Free and Forced Vibrations of a Closed Elastic Spherical Shell Fixed to an Equatorial Beam—Part I: The Governing Equations and Special Solutions

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The classical theories of shells and curved beams are used to develop the equations of motion of an elastically isotropic spherical shell attached to an elastic equatorial beam of rectangular cross section. The mass densities and elasticities of the shell and beam are, in general, different. Remarkably, for the natural frequencies, the final set of eight linear homogeneous algebraic equations uncouples into two sets of four. The only approximations made are of the same order of magnitude as those inherent in classical shell and beam theory. Although solutions of the shell equations involve Legendre functions (and not polynomials), the final set of algebraic equations involve only trigonometric and gamma functions. Several special exact solutions are given. In Part II, perturbation techniques are used to find the natural frequencies for beam-shell configurations ranging from nearly pure beams to nearly pure shells. [DOI: 10.1115/1.3197466]

1 Introduction

This paper was motivated by the question: Is it possible to image living tissue by injecting into veins or arteries minute spherical bubbles engirdled by copper rings and then exciting these ring-bubble configurations by an external electromagnetic field? As the dimensions of the ring-bubbles are several orders of magnitude larger than a typical molecular diameter, a reasonable approach to this question is via a continuum model. Thus, we assume that the spherical bubble and the encircling wire may be modeled by *classical shell and beam* theory. By "classical" we mean that we neglect the effects of transverse shear stress on the stress-strain relations and rotary inertia on the equations of motion. Within the limits of classical theory, the effect of the beam on the shell is that of a dynamic line load acting at the equator of the midsurface of the shell. To find the overall response requires that we first find the general solutions of the equations of motion of two hemispherical shells and then join these at the equator by four kinematic continuity conditions and four kinetic jump conditions, the latter coming from the differential equations of motion satisfied by the attached circular beam. Natural frequencies and mode shapes are found, which can be used to synthesize the response of the ring-sphere to an external (piezoelectric) force applied to the ring.

The linearized equations of motion of an elastically isotropic shell have been developed, for example, by Niordson in Ref. [1], where natural frequencies and mode shapes have been worked out analytically (in terms of Legendre polynomials) for complete shells and numerically for various open shells. (See also, Ref. [2], p. 549, where earlier results by Lamb are summarized.) We use this work in Part I of this paper. As Niordson [1] showed, the tensor form of the equations of motion of the shell plus the special symmetry of a sphere lead straightaway to three reduced scalar equations. For the beam equations we use the well-known linearized equations of motion of a circular beam of rectangular cross section subject to external forces and a moment, these being the reaction forces and moment exerted by the shell and the external electromagnetic forces. Then, introducing spherical coordinates, we write down explicitly the eight aforementioned junction conditions at the equator. Remarkably, the equations for the natural frequencies of the beam-shell uncouple into two classes, which simplifies the subsequent analysis considerably. To end Part I, we compute exactly the natural frequencies for several special cases.

In Part II, we take advantage of the small thickness to radius ratios of the shell and the beam to compute approximations to the natural frequencies and modes of the coupled beam-shell system.

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2 The Equations of Motion of an Elastically Isotropic Spherical Shell

The geometry of a sphere makes tensor analysis particularly convenient for deriving scalar equations of motion for an elastically isotropic spherical shell of midsurface radius R. Thus, let the position of a point on the midsurface with respect to fixed dextral Cartesian axes $Oxyz$ be denoted by $\mathbf{R}=R\mathbf{n}(\theta^\alpha)$, where $\mathbf{n}$ is the *outward* unit normal at a point on the midsurface with Gaussian coordinates $\theta^\alpha, \alpha=1,2$.With a comma followed by a subscript denoting partial differentiation with respect to the coordinate carrying that index, the covariant base vectors are

$$\mathbf{R}_{,\alpha} = R\mathbf{n}_{,\alpha}, \alpha = 1,2 \tag{1}$$

The Christoffel symbols and the covariant components of the surface curvature tensor are defined by the second derivative of the position

$$R\mathbf{n}_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{R}_{,\gamma} + b_{\alpha\beta}\mathbf{n}, \tag{2}$$

where a repeated index is to be summed from 1 to 2. Denoting the covariant components of the metric tensor by $a_{\alpha\beta} \equiv R^2\mathbf{n}_{,\alpha}\bullet\mathbf{n}_{,\beta}$, we see from Eqs. (1) and (2) that

$$b_{\alpha\beta} = R\mathbf{n}\bullet\mathbf{n}_{,\alpha\beta} = -R\mathbf{n}_{,\alpha}\bullet\mathbf{n}_{,\beta} = -R^{-1}a_{\alpha\beta} \tag{3}$$

It is this geometrical fact that makes a spherical surface so special. As is standard, indices are raised or lowered with respect to $a_{\alpha\beta}$ or $a^{\alpha\beta}$, where $a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma$, the Kronecker delta.

The classical shell theory developed independently by Sanders [3] and Koiter [4], and dubbed by Budiansky and Sanders [5] as the "best" in a family of competing tensorial theories, is distinguished by a bending strain tensor that is expressible in terms of midsurface rotations only. However, as Niordson points out [1], one of the competing tensorial theories is actually simpler for the exceptional case of a spherical shell.

Thus, with $N^{\alpha\beta}$, $M^{\alpha\beta}$, $K^{\alpha\beta}$, and $E^{\alpha\beta}$ denoting, respectively, the contravariant components of the modified symmetric stress resultant, stress couple, bending strain, and extensional strain tensors used by Niordson [1], and with

$$\mathbf{u}=u^{\alpha}\mathbf{R},_{\alpha}+w\mathbf{n}, \quad \mathbf{p}=p^{\alpha}\mathbf{R},_{\alpha}+p\mathbf{n}, \quad \mathbf{m}=m^{\alpha}\mathbf{R},_{\alpha}\times\mathbf{n} \tag{4}$$

denoting, respectively, the midsurface displacement, the external pressure, and the external moment; the equations of motion and compatibility conditions read

$$N^{\alpha\beta}|_{\alpha}+p^{\beta}-R^{-1}m^{\beta}=\rho_s h\ddot{u}^{\beta} \tag{5}$$

$$-M^{\alpha\beta}|_{\alpha\beta}-R^{-2}M-R^{-1}N+p-m^{\alpha}|_{\alpha}=\rho_s h\ddot{w} \tag{6}$$

$$\hat{K}^{\alpha\beta}|_{\alpha}=0 \tag{7}$$

$$\hat{E}^{\alpha\beta}|_{\alpha\beta}+R^{-2}E-R^{-1}K=0 \tag{8}$$

In these equations, a vertical bar denotes covariant differentiation (based on $a_{\alpha\beta}$), a superimposed double dot denotes the second time derivative (acceleration), ρ_s is the constant mass density of the shell, h is the constant shell thickness, $N\equiv N^{\alpha}_{\alpha}$, etc., is a trace, and

$$\hat{K}^{\alpha\beta}\triangleq a^{\alpha\beta}K-K^{\beta\alpha}, \text{etc.} \tag{9}$$

The *field equations* (5)–(8) are supplemented by the uncoupled, elastically isotropic *constitutive relations*

$$\hat{E}^{\alpha\beta}=A[a^{\alpha\beta}N-(1+\nu)N^{\alpha\beta}] \tag{10}$$

$$M^{\alpha\beta}=D[a^{\alpha\beta}K-(1-\mu)\hat{K}^{\alpha\beta}] \tag{11}$$

where A and D are elastic constants and ν and μ are, respectively, Poisson ratios of stretching and bending. Conventionally

$$A=\frac{1}{E_S h}, \quad D=\frac{E_S h^3}{12(1-\nu^2)}, \quad \nu=\mu \tag{12}$$

where E_S is the Young's modulus of the shell. However, we shall take the four elastic constants as independent until Part II. Note that the *homogeneous form of* Eqs. (5)–(8), (10), and (11) exhibit the *static-geometric duality*

$$\{N^{\alpha\beta}:\hat{K}^{\alpha\beta}, M^{\alpha\beta}:-\hat{E}^{\alpha\beta}, A:-D, \nu:-\mu\} \tag{13}$$

The meaning of this symbol is that if the variables and constants to the left of the colons are replaced by those on the right, then Eqs. (5), (6), and (10) *with the external loads and inertial terms set to zero*, turn into Eqs. (7), (8), and (11).

To complete our field equations we add the strain-displacement relations

$$E_{\alpha\beta}=\tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})+R^{-1}a_{\alpha\beta}w, \quad K_{\alpha\beta}=w_{|\alpha\beta}+R^{-2}a_{\alpha\beta}w \tag{14}$$

Finally, we follow Niordson [1] and decompose the tangential displacements by setting

$$u_{\alpha}=R(\psi,_{\alpha}+\varepsilon_{\alpha\beta}\chi|^{\beta}) \tag{15}$$

so that

$$E_{\alpha\beta}=R\left[\psi|_{\alpha\beta}+\tfrac{1}{2}(\varepsilon_{\alpha\gamma}\chi|^{\gamma}_{\beta}+\varepsilon_{\beta\gamma}\chi|^{\gamma}_{\alpha})\right]+R^{-1}a_{\alpha\beta}w \tag{16}$$

In Eq. (15), $\varepsilon_{\alpha\beta}$ is the covariant components of the surface permutation tensor.

3 Harmonic Motion of Hemispherical Shells

If we assume that all dependent variables have the form $f(\theta^{\alpha},t)=\tilde{f}(\theta^{\alpha})e^{i\omega t}$, drop the tildes, and take the real part, the field equations retain their forms except that $\ddot{\psi}\rightarrow-\omega^2\psi$, etc. Furthermore, because the external loads on the closed spherical shell are taken as line loads acting at the equator, we may drop these terms when considering the field equations in either of the hemispheres $\mathbf{n}\cdot\mathbf{e}_z>0$ or $\mathbf{n}\cdot\mathbf{e}_z<0$, where $\mathbf{e}_z$ is a unit vector along the z-axis. Following Ref. [1], we now reduce the field equations in each hemisphere to two coupled equations for w and ψ and a single equation for χ.

First, we invert the constitutive relation (10) to obtain

$$A(1-\nu^2)N^{\alpha\beta}=\nu a^{\alpha\beta}E+(1-\nu)E^{\alpha\beta}=a^{\alpha\beta}E-(1-\nu)\hat{E}^{\alpha\beta} \tag{17}$$

Then, we insert the constitutive relations (11) and (17) into Eq. (6) and note the compatibility condition (7) to obtain

$$-DR^{-2}(\Delta+1+\mu)K-[AR(1-\nu)]^{-1}E+\rho_s h\omega^2 w=0 \tag{18}$$

where $\Delta(\,)\triangleq R^2a^{\alpha\beta}(\,)|_{\alpha\beta}$ is the dimensionless Laplacian on the spherical midsurface. But from Eqs. $(14)_2$ and (16)

$$K=R^{-2}(\Delta+2)w, \quad E=R^{-1}(\Delta\psi+2w) \tag{19}$$

Thus, Eq. (18) reduces to the first of our two coupled equations

$$\varepsilon^2(\Delta+2)(\Delta+1+\mu)w+(1-\nu)^{-1}(\Delta\psi+2w)-\Omega^2 w=0 \tag{20}$$

where

$$\varepsilon^2\triangleq\frac{AD}{R^2}=\frac{h^2}{12(1-\nu^2)R^2}\ll 1 \quad \text{and} \quad \Omega^2\triangleq\rho_s hAR^2\omega^2 \tag{21}$$

To obtain a second equation relating the normal displacement w and the displacement function ψ, we first differentiate the (dynamic) equilibrium equation (5) and insert the decomposition (15) to obtain

$$N^{\alpha\beta}|_{\alpha\beta}+\rho_s hR^{-1}\omega^2\Delta\psi=0 \tag{22}$$

But by Eqs. (8), (17), and (19)

$$A(1-\nu^2)R^3N^{\alpha\beta}|_{\alpha\beta}=R[\Delta E+(1-\nu)(E-RK)]=\Delta[\Delta\psi+(1-\nu)\psi+(1+\nu)w] \tag{23}$$

Hence, by Eq. $(21)_2$, Eq. (22) reduces to

$$\Delta[\Delta\psi+(1-\nu)\psi+(1+\nu)w+(1-\nu^2)\Omega^2\psi]=0 \tag{24}$$

The term in brackets is harmonic but must vanish, as Niordson shows [1]. Thus, the equation that complements Eq. (20) is

$$\Delta\psi+(1-\nu)\psi+(1+\nu)w+(1-\nu^2)\Omega^2\psi=0 \tag{25}$$

Finally, to derive an equation for the torsionlike displacement potential χ, we first apply the operator $\varepsilon_{\beta\gamma}(\,)|^{\gamma}$ to both sides of Eq. (5) and note Eq. (15), the decomposition of the tangential displacement. Thus

$$\varepsilon_{\beta\gamma}N^{\alpha\beta}|^{\gamma}_{\alpha}+\rho R^{-2}\omega^2\Delta\chi=0 \tag{26}$$

Next, from Eq. (16) and with the aid of the identity $\varepsilon^{\beta\gamma}\varepsilon_{\alpha\lambda}\equiv\delta^{\beta}_{\alpha}\delta^{\gamma}_{\lambda}-\delta^{\beta}_{\lambda}\delta^{\gamma}_{\alpha}$

$$\varepsilon^{\beta\gamma}E_{\alpha\beta}|^{\alpha}=R^{-1}\varepsilon^{\beta\gamma}(\Delta\psi+w),_{\beta}+\chi|^{\gamma\alpha}_{\alpha}-\tfrac{1}{2}R^{-2}\Delta\chi|^{\gamma} \tag{27}$$

Now, because covariant differentiation does not commute on a curved surface,

$$\chi|^{\gamma\alpha}_{\alpha}=\chi|^{\alpha\gamma}_{\alpha}+G\chi|^{\gamma}=R^{-2}(\Delta+1)\chi|^{\gamma} \quad \text{on a sphere} \tag{28}$$

where the Gaussian curvature $G=R^{-2}$. Thus, Eq. (27) implies that

$$\varepsilon^{\beta\gamma}E_{\alpha\beta}|^{\alpha}_{\gamma}=\tfrac{1}{2}R^{-4}\Delta(\Delta+2)\chi \tag{29}$$

Upon inserting Eqs. (17) and (29) into Eq. (26) and noting Eq. $(21)_2$, we have

$$\Delta[\Delta + 2 + 2(1+\nu)\Omega^2]\chi = 0 \tag{30}$$

As Niordson shows [1], the harmonic term in brackets vanishes. Thus, the third equation governing the time-harmonic motion of either hemisphere of a closed spherical shell is

$$[\Delta + 2 + 2(1+\nu)\Omega^2]\chi = 0 \tag{31}$$

4 Spherical Coordinates and Jump Conditions at the Equator

We now introduce spherical coordinates, $\theta^1=\theta$, $0\le\theta<2\pi$, $\theta^2=\phi$, $0\le\phi\le\pi$. Continuity of displacement and rotation at the equator requires that

$$[\![\mathbf{u}]\!] = \mathbf{0}, \quad [\![w,_\phi]\!] = 0, \tag{32}$$

where the symbol $[\![\bullet]\!]$ denotes the jump in a function as the polar angle ϕ increases through $1/2\pi$. Because $w|_{\alpha\beta}=w,_{\alpha\beta}-\Gamma^{\gamma}_{\alpha\beta}w,_\gamma$ and $\Delta w=w,_{\theta\theta}+\csc\phi(\sin\phi w,_\phi),_\phi$, the kinematic continuity (or jump) condition (32), the constitutive relation (11), the compatibility condition (7), plus the strain-displacement relation (14) imply that

$$[\![M_{12}]\!] = 0, \quad [\![M_{22}]\!] = D[\![\Delta w]\!], \quad M^{\alpha\beta}|_\alpha = DR^{-4}\Delta w|^\beta \tag{33}$$

which, in turn, imply that at the equator, in addition to a line load, the beam exerts a line moment on the shell (and vice versa) that lies along the circumferential unit vector $\mathbf{e}_\theta$.

The equations of motion (5) and (6), with the $e^{i\omega t}$ dependence of each term removed, can be written as

$$\frac{1}{\sqrt{a}}\frac{\partial}{\partial\theta^\alpha}(\sqrt{a}\mathbf{N}^\alpha) + \mathbf{p} + \rho_s h\omega^2\mathbf{u} = \mathbf{0} \tag{34}$$

where $a=\det(a_{\alpha\beta})$ and

$$\mathbf{N}^\alpha = (N^{\alpha\beta} + R^{-1}M^{\alpha\beta})\mathbf{R},_\beta + Q^\alpha\mathbf{n}, \quad Q^\alpha \triangleq -(M^{\alpha\beta}|_\beta + m^\alpha) \tag{35}$$

We now set

$$\mathbf{e}_r = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y = \mathbf{n}\left(\theta,\tfrac{1}{2}\pi\right), \quad \mathbf{e}_\theta = -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y \tag{36}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the standard orthogonal base vectors associated with the coordinate system $Oxyz$, and take $\{\mathbf{p}, m^\alpha\mathbf{R},_\alpha\} = R^{-1}\{\mathbf{P}(\theta), M(\theta)\mathbf{e}_z(\theta)\}\delta(\phi-1/2\pi)$, where δ is the Dirac delta function. Thus, Eq. (34) expands to

$$\frac{\partial\mathbf{N}^1}{\partial\theta} + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\mathbf{N}^2) + R^{-1}\mathbf{P}\delta\left(\phi - \frac{1}{2}\pi\right) + \rho_S\omega^2\mathbf{u} = \mathbf{0} \tag{37}$$

Integrating both sides of Eq. (37) with respect to ϕ from $1/2\pi - a$ to $1/2\pi+a$ and letting $a\to 0$, we obtain the *jump condition*

$$R[\![\mathbf{N}^2]\!] + \mathbf{P}(\theta) = \mathbf{0} \tag{38}$$

Note that Eqs. $(33)_3$ and $(35)_2$ imply that $-[\![Q^\alpha]\!]=[\![M^{2\beta}|_\beta]\!] = DR^{-4}[\![(\Delta w),_\phi]\!]$. Furthermore, integrating Eq. $(35)_2$ from $\phi = 1/2\pi - a$ to $\phi=\frac{1}{2}\pi+a$, letting $a\to 0$ and assuming that the transverse shear stress resultant is finite, we obtain by Eq. (33) the additional jump condition

$$[\![M^{22}]\!] = -R^{-2}M(\theta) \tag{39}$$

so that Eq. (38) expands to

$$R^2([\![N^{21}]\!]\mathbf{e}_\theta + [\![N^{22} - R^{-3}M(\theta)]\!]\mathbf{e}_z - DR^{-5}[\![(\Delta w),_\phi]\!]\mathbf{e}_r) + \mathbf{P}(\theta) = \mathbf{0} \tag{40}$$

Equations (32), (39), and (40) are equivalent to *eight* scalar jump conditions.

5 The Equatorial Beam Equations

The classical linear dynamic equations of a circular beam of rectangular cross section comprise three equations of translational momentum and three of rotational momentum, with rotary inertia neglected. Recalling that all dependent variables vary as $e^{i\omega t}$, we set

$$\mathbf{U}(\theta) \triangleq \mathbf{u}\left(\theta,\tfrac{1}{2}\pi\right) \triangleq U(\theta)\mathbf{e}_\theta(\theta) + V(\theta)\mathbf{e}_z + W(\theta)\mathbf{e}_r(\theta) \tag{41}$$

and

$$\Phi(\theta) \triangleq R^{-1}\varphi_2\left(\theta,\tfrac{1}{2}\pi\right) = R^{-1}[\Lambda(\theta) - V(\theta)], \quad \Lambda \triangleq w,_\phi\left(\theta,\tfrac{1}{2}\pi\right) \tag{42}$$

where Λ is the *local angle of twist* of the beam.

With

$$\mathbf{T} = T_\theta\mathbf{e}_\theta + T_z\mathbf{e}_z + T_r\mathbf{e}_r \quad \text{and} \quad \mathbf{C} = C_\theta\mathbf{e}_\theta + C_z\mathbf{e}_z + C_r\mathbf{e}_r \tag{43}$$

denoting, respectively, the force and couple acting over a section θ=constant, the equations of harmonic motion of a beam of rectangular cross section H (in the z-direction) $\times B$ (in the radial direction) and mass density ρ read

$$R^{-1}\mathbf{T}' + \rho BH\omega^2\mathbf{U} + \mathbf{F}(\theta) = \mathbf{P}(\theta), \quad R^{-1}\mathbf{C}' + \mathbf{e}_\theta\times\mathbf{T} = M(\theta)\mathbf{e}_\theta(\theta) \tag{44}$$

where $(\ )' = d/d\theta$ and $\mathbf{F}(\theta)$ is the external force/length applied to the beam, the $e^{i\omega t}$ dependence being assumed. From Eqs. (43) and $(44)_2$,

$$\mathbf{T} = T_\theta\mathbf{e}_\theta + R^{-1}[(C_\theta - C_r')\mathbf{e}_z + C_z'\mathbf{e}_r], \quad C_\theta' + C_r = RM(\theta) \tag{45}$$

Hence, by Eqs. (41) and $(44)_1$, the jump condition (40) reduces to

$$R^2([\![N^{21}]\!]\mathbf{e}_\theta + [\![N^{22} - R^{-3}M(\theta)]\!]\mathbf{e}_z - DR^{-5}[\![(\Delta w),_\phi]\!]\mathbf{e}_r) + R^{-1}\{T_\theta\mathbf{e}_\theta + R^{-1}[(C_\theta - C_r')\mathbf{e}_z + C_z'\mathbf{e}_r]\}' + \rho BH\omega^2\mathbf{U} + \mathbf{F}(\theta) = \mathbf{0} \tag{46}$$

The system of equations—shell plus attached equatorial beam—is complete upon the addition of constitutive relations for the beam for which we first need expressions for the extensional and bending strains in terns of $\mathbf{U},\Phi$, and their derivatives. By Eqs. (41), (42), and (45), the internal work (IW) done by the force and couples in the beam is

$$\begin{aligned} IW &= -\int_0^{2\pi}[\mathbf{T}'\bullet\mathbf{U} + (C_\theta' + C_r)\Phi]d\theta \\ &= -\int_0^{2\pi}\{\bar{T}_\theta' U + R^{-1}[(C_\theta' - C_r'')V + (C_z'' + C_z - \bar{T}_\theta)W] \\ &\quad + (C_\theta' + C_r)\Phi\}d\theta \end{aligned} \tag{47}$$

where $\bar{T}_\theta \triangleq T_\theta + R^{-1}C_z$. Integration by parts to remove all derivatives on the force and couples yields

$$IW = \int_0^{2\pi}(R\bar{T}_\theta E_\theta + C_\theta K_\theta + C_z K_z + C_r K_r)d\theta \tag{48}$$

where the strains associated with the force $\bar{T}_\theta$ and the couples C_θ, C_z, and C_r are

$$E_\theta = R^{-1}(U' + W), \quad K_\theta = R^{-1}\Lambda', \quad K_z = -R^{-1}\mathcal{L}W, \quad K_r = R^{-1}(\mathcal{L}V - \Lambda) \tag{49}$$

Here

$$\mathcal{L} \triangleq d^2/d\theta^2 + 1 \tag{50}$$

In a linear theory free of initial forces and couples, the strain-energy density of the beam is a homogeneous quadratic function of the strains. For simplicity, we assume no cross coupling so that the constitutive equations take the form

$$\bar{T}_\theta = EBHR^{-1}(U' + W), \quad C_\theta = (GJB^2H^2/R^2)\Lambda' \tag{51}$$

$$C_z = -(EB^3H/12R^2)\mathcal{L}W, \quad C_r = (EH^3B/12R^2)(\mathcal{L}V - \Lambda) \tag{52}$$

where $2(1+\nu)G \triangleq E$, E is Young's modulus of the beam, and J is a geometric torsion factor. See Eq. (171) on p. 313 of Ref. [6]. For a square cross section, $J=0.1406\ldots$.

6 The Jump Conditions in Terms of the Scalars ψ, χ, and w

From Eq. (32), the third and fourth of the eight scalar jump conditions are $[\![w]\!]=[\![w,_\phi]\!]=0$. The first and second of the jump condition (32) follow upon using Eq. (15) to express $[\![u_\alpha]\!]=0$ in terms of ψ and χ. Thus, noting that at $\phi=1/2\pi$, $\varepsilon_{12}=-\varepsilon_{21}=\sqrt{a}=R^2$, and $\chi|^1=R^{-2}\chi,_\theta$, $\chi|^2=R^{-2}\chi,_\phi$, we have, in summary, four homogeneous jump conditions

$$[\![\psi,_\theta + \chi,_\phi]\!] = [\![\psi,_\phi - \chi,_\theta]\!] = [\![w]\!] = [\![w,_\phi]\!] = 0 \tag{53}$$

plus four nonhomogeneous scalar jump conditions that follow from Eqs. $(45)_2$ and (46) and which we now proceed to express in terms of our three basic unknowns, w, ψ, and χ.

First note by Eqs. $(33)_2$, (39), $(51)_2$, and $(52)_2$, that the moment equilibrium equation, Eq. $(45)_2$, becomes

$$(GJB^2H^2/R^2)\Lambda'' + (EH^3B/12R^2)(\mathcal{L}V - \Lambda) = -R^{-1}D[\![\Delta w]\!] \tag{54}$$

Second, because $\chi|_1^1=R^{-2}\chi,_{\theta\theta}$ and $\chi|_2^2=R^{-2}\chi,_{\phi\phi}$ at $\phi=1/2\pi$ and because Eq. $(53)_4$ implies that $[\![\chi,_{\theta\theta}]\!]=[\![\psi,_{\theta\phi}]\!]$, we have from Eqs. (16) and (17)

$$[\![N^{21}]\!] = [2AR^3(1+\nu)]^{-1}[\![\Delta\chi]\!] \tag{55}$$

Finally, because $[\![w]\!]=0$, it is not difficult to show by Eq. $(53)_2$ that $[\![E_{11}]\!]=R[\![\psi,_{\theta\theta}+\chi,_{\theta\phi}]\!]=0$ and $[\![E_{22}]\!]=R[\![\psi,_{\phi\phi}-\chi,_{\theta\phi}]\!]=R[\![\Delta\psi]\!]$. Hence, from the constitutive relation (17)

$$[\![N^{22}]\!] = [AR^3(1-\nu^2)]^{-1}[\![\Delta\psi]\!] \tag{56}$$

If we resolve the external force acting on the ring-beam into components by setting

$$\mathbf{F} = F_\theta(\theta)\mathbf{e}_\theta(\theta) + F_z(\theta)\mathbf{e}_z + F_r\mathbf{e}_r(\theta) \tag{57}$$

and use the $\mathbf{e}_r$-component of the equation of motion to eliminate $\bar{T}_\theta$ from the $\mathbf{e}_\theta$-component, then the three scalar jump conditions implied by Eqs. (46), with the aid of Eqs. (51), (52), and (55)–(57), read

$$-DR^{-3}[\![(\Delta w),_\phi]\!]' + [2AR(1+\nu)]^{-1}[\![\Delta\chi]\!] - (EB^3H/12R^4)\mathcal{L}^2W' + \rho BH\omega^2(U+W') + F_\theta(\theta) + F_r'(\theta) = 0 \tag{58}$$

$$[AR(1-\nu^2)]^{-1}[\![\Delta\psi]\!] - (EBH^3/12R^4)\mathcal{L}(\mathcal{L}V - \Lambda) + \rho BH\omega^2V + F_z(\theta) = 0 \tag{59}$$

$$-DR^{-3}[\![(\Delta w),_\phi]\!] - (EBH/R^2)(U'+W) - (EB^3H/12R^4)\mathcal{L}^2W + \rho BH\omega^2W + F_r(\theta) = 0 \tag{60}$$

Note by Eqs. (25) and (31) that because $[\![w]\!]=0$

$$[\![\Delta\psi]\!] = -(1-\nu)[1+(1+\nu)\Omega^2][\![\psi]\!] \quad \text{and}$$

$$[\![\Delta\chi]\!] = -2[1+(1+\nu)\Omega^2][\![\chi]\!] \tag{61}$$

7 Two Classes of Free Vibrations

At the equator ($\phi=1/2\pi$), the four homogeneous jump conditions (Eq. (53)) must be supplemented by the following four equations for computing the beam displacements and local rotation in terms of w, ψ, and χ:

$$2U = \langle\psi\rangle' + \langle\chi,_\phi\rangle, \quad 2V = \langle\psi,_\phi\rangle - \langle\chi\rangle', \quad 2W = \langle w\rangle, \quad 2\Lambda = \langle w,_\phi\rangle \tag{62}$$

where $\langle\psi\rangle \triangleq \psi(1/2\pi+,\theta) + \psi(1/2\pi-,\theta)$, etc.

Remarkably, the homogeneous forms of Eqs. (58) and (60) uncouple from the homogeneous forms of Eqs. (54) and (59) yielding two distinct classes of free vibrations characterized by the following.

For Class I,

$$V = \Lambda = [\![\psi]\!] = [\![\Delta w]\!] = 0 \tag{63}$$

For Class II,

$$U = W = [\![\chi]\!] = [\![(\Delta w),_\phi]\!] = 0 \tag{64}$$

We give more details in Sec. 10.

8 Fourier Decomposition

We assume that the external force acting on the beam has the representation

$$\mathbf{F}(\theta) = \mathcal{R}\sum_{m=-M}^{M}\mathbf{F}_m e^{im\theta} \tag{65}$$

where $\mathcal{R}$ denotes "the real part of." The nonhomogeneous terms in the boundary conditions (54) and (58)–(60) induce similar representations for w, ψ, and χ. Because of linearity, we may consider separately the equations satisfied by each Fourier component $\mathcal{R}\{w_m,\psi_m,i\chi_m\}e^{im\theta}$.

Let

$$\Delta_m \triangleq \frac{d^2}{d\phi^2} + \cot\phi\frac{d}{d\phi} - m^2\csc^2\phi = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] - \frac{m^2}{1-x^2} \tag{66}$$

where $x=\cos\phi$. Then the three field Eqs. (20), (25), and (31) reduce to

$$\mathcal{B}(\varepsilon^2)\{w_m,\psi_m\} \triangleq \varepsilon^2(\Delta_m+2)(\Delta_m+1+\mu)w_m + (1-\nu)^{-1}(\Delta_m\psi_m + 2w_m) - \Omega^2w_m = 0 \tag{67}$$

$$\mathcal{D}\{w_m,\psi_m\} \triangleq \Delta_m\psi_m + (1-\nu)\psi_m + (1+\nu)w_m + (1-\nu^2)\Omega^2\psi_m = 0 \tag{68}$$

and

$$[\Delta_m + 2 + 2(1+\nu)\Omega^2]\chi_m = 0 \tag{69}$$

Now

$$\Delta_m P_\sigma^m(x) = -\lambda P_\sigma^m(x), \quad \lambda \triangleq \sigma(\sigma+1) \tag{70}$$

where P_σ^m is *the associated Legendre function of degree* σ and order m of the first kind that may be expressed in terms of the hypergeometric function $F(a,b;c;z)$ as

$$P_\sigma^m(x) = \frac{(-2)^{-m}\Gamma(\sigma+m+1)}{m!\,\Gamma(\sigma-m+1)}(1-x^2)^{1/2m}F\left(m+1+\sigma, m-\sigma; m+1; \frac{1}{2}-\frac{1}{2}x\right) \tag{71}$$

where Γ is the gamma function. See p. 174 of Ref. [7]. In particular, when we come to impose the jump conditions at the equator ($x=0$), we shall need the values there of P_σ^m and its first derivative. Thus, from pp. 2 and 171 of Ref. [7]

$$P_\sigma^m(0)=\frac{2^m}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma+\frac{1}{2}m\right)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}m\right)}\cos\left[\tfrac{1}{2}\pi(\sigma+m)\right]$$
$$=\frac{2^m\sqrt{\pi}}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\sigma-\frac{1}{2}m\right)}\triangleq P(\sigma,m) \quad (72)$$

and

$$-\left.\frac{dP_\sigma^m(\cos\phi)}{d\phi}\right|_{\phi=1/2\pi}=\left.\frac{dP_\sigma^m(-\cos\phi)}{d\phi}\right|_{\phi=1/2\pi}=\left.\frac{dP_\sigma^m(x)}{dx}\right|_{x=0}$$
$$=\frac{2^{m+1}}{\sqrt{\pi}}\frac{\Gamma\left(1+\frac{1}{2}\sigma+\frac{1}{2}m\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma-\frac{1}{2}m\right)}\sin\left[\tfrac{1}{2}\pi(\sigma+m)\right]$$
$$=\frac{2^{m+1}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma-\frac{1}{2}m\right)\Gamma\left(1-\frac{1}{2}\sigma-\frac{1}{2}m\right)}$$
$$\triangleq S(\sigma,m) \quad (73)$$

Because $P_\sigma^m(x)$ is singular at $x=-1$, we work with $P_\sigma^m(x)$ in the lower hemispherical shell and $P_\sigma^m(-x)$ in the upper hemispherical shell. We must discard a second independent solution of Eq. (70), usually denoted by $Q_\sigma^m(x)$, because it is singular at *both* $x=\pm 1$.

We now seek solutions of the form

$$\{w_m,\psi_m\}=\{\hat{w}_m^\pm,\hat{\psi}_m^\pm\}P_\sigma^m(\mp x),\quad \chi_m=\hat{\chi}_m^\pm P_\tau^m(\mp x) \quad (74)$$

the lower sign indicating coefficients associated with the lower $(0<x<1)$ hemisphere and the upper sign with the upper $(-1<x<0)$ hemisphere. Substituting Eq. $(74)_2$ into Eq. (69) and noting Eq. (70), we obtain

$$\tau(\tau+1)=2[1+(1+\nu)\Omega^2],\quad \tau>0 \quad (75)$$

The substitution of Eq. $(74)_1$ into Eqs. (67) and (68) yields two linear homogeneous equations for the constants $\hat{w}_m^\pm$ and $\hat{\psi}_m^\pm$ provided that

$$\begin{vmatrix}\varepsilon^2(2-\lambda)(1+\mu-\lambda)+2(1-\nu)^{-1}-\Omega^2 & -(1-\nu)^{-1}\lambda \\ (1+\nu) & 1-\nu-\lambda+(1-\nu^2)\Omega^2\end{vmatrix}=0 \quad (76)$$

This equation yields three values, λ_1, λ_2, and λ_3, each depending on the Poisson ratios of stretching and bending, ν and μ, and the square of the dimensionless frequency, Ω^2. Associated with each λ_i is an unknown constant ${}_i\hat{w}_m^\pm$ and, from the second line of Eq. (76), a constant

$${}_i\hat{\psi}_m^\pm=\frac{(1+\nu)_i\hat{w}_m^\pm}{\lambda_i-(1-\nu)[1+(1+\nu)\Omega^2]}\triangleq\hat{C}(\nu,\lambda_i,\Omega^2)_i\hat{w}_m^\pm \quad (77)$$

Thus,

$$w_m^\pm=\sum_{i=1}^3{}_i\hat{w}_m^\pm P_{\sigma_i}^m(\mp x),\quad \psi_m^\pm=\sum_{i=1}^3\hat{C}(\nu,\lambda_i,\Omega^2)_i\hat{w}_m^\pm P_{\sigma_i}^m(\mp x),$$
$$\lambda_i=\sigma_i(\sigma_i+1) \quad (78)$$

9 Imposition of the Jump (Boundary) Conditions

If we use Eq. (61) and introduce the dimensionless constants

$$g\triangleq(\rho/\rho_S)(BH/hR),\quad k\triangleq(E/E_S)(BH/hR) \quad (79)$$

in addition to ε^2 and Ω^2 defined in Eq. (21), then the Fourier components of the three jump conditions (58)–(60) can be written

$$\underline{-\varepsilon^2m[\![(\Delta_mw_m)_{,\phi}]\!]}-[(1+\nu)^{-1}+\Omega^2][\![\chi_m]\!]-k(B^2/12R^2)m(m^2-1)^2W_m+g\Omega^2(U_m+mW_m)+AR(mF_r^m-F_\theta^m)=0 \quad (80)$$

$$-[(1+\nu)^{-1}+\Omega^2][\![\psi_m]\!]-k(H^2/12R^2)(m^2-1)[(m^2-1)V_m+\Lambda_m]+g\Omega^2V_m+ARF_z^m=0 \quad (81)$$

$$-\varepsilon^2[\![(\Delta_mw_m)_{,\phi}]\!]+k(mU_m-W_m)-\underline{k(B^2/12R^2)(m^2-1)^2W_m}+g\Omega^2W_m+ARF_r^m=0 \quad (82)$$

where Δ_m is defined in Eq. (66), and the underlined terms will prove to be negligible. Henceforth, we assume that $m=\mathrm{O}(1)$. Then, Eq. (68) implies that ψ_m and w_m are of comparable magnitude, as well as V_m and Λ_m. The Fourier components of the fourth jump condition on moments, Eq. (54), nondimensionalized, read

$$k(GJ/E)(BH/R^2)m^2\Lambda_m+k(H^2/12R^2)[(m^2-1)V_m+\Lambda_m]+\varepsilon^2[\![\Delta_mw_m]\!]=0 \quad (83)$$

10 The Fourier Components of the Two Classes of Free Vibrations

For *Class I* vibrations, the conditions $[\![\psi_m]\!]=[\![w_m]\!]=[\![\Delta w_m]\!]=0$ plus Eqs. (72), (73), and (78) imply that $[\![{}_i\hat{w}_m]\!]=0$, $i=1,2,3$. Hence, ${}_i\hat{w}_m^+={}_i\hat{w}_m^-\triangleq\hat{w}_m^i$ so that

$$[\![w_{m,\phi}]\!]=0\Rightarrow\sum_{i=1}^3\hat{w}_m^iS(\sigma_i,m)=0\quad\text{and}$$

$$[\![(\Delta w_m)_{,\phi}]\!]=2\sum_{i=1}^3\hat{w}_m^i\lambda_iS(\sigma_i,m) \quad (84)$$

Moreover, because $\langle\psi_{m,\phi}\rangle=2\Sigma_{i=1}^3\hat{w}_m^i\hat{C}(\nu,\lambda_i,\Omega^2)S(\sigma_i,m)=0$

$$V_m=0\Rightarrow\langle\chi_m\rangle=0\Rightarrow\hat{\chi}_m^+=-\hat{\chi}_m^-\triangleq\hat{\chi}_m \quad (85)$$

Thus, Eqs. $(62)_{1,3}$, (72), (74), (78), and (85) yield

$$U_m=\frac{1}{2}m\langle\psi_m\rangle+\frac{1}{2}\langle\chi_{m,\phi}\rangle=m\sum_{i=1}^3\hat{w}_m^i\hat{C}(\nu,\lambda_i,\Omega^2)P(\sigma_i,m)+\hat{\chi}_mS(\tau,m) \quad (86)$$

and

$$W_m=\sum_{i=1}^3\hat{w}_m^iP(\sigma_i,m),\quad [\![\chi_m]\!]=2\hat{\chi}_mP(\tau,m) \quad (87)$$

Finally, the jump condition $[\![v_m]\!]=[\![\psi_{m,\phi}]\!]+m[\![\chi_m]\!]=0$ becomes

$$\sum_{i=1}^3\hat{w}_m^i\hat{C}(\nu,\lambda_i,\Omega^2)S(\sigma_i,m)+m\hat{\chi}_mP(\tau,m)=0 \quad (88)$$

When Eqs. $(84)_2$, (86) and (87) are substituted into Eqs. (80) and (82), we obtain, together with Eqs. $(84)_1$ and (88), four homogeneous linear algebraic equation for the four constants $\hat{w}_m^i$, $i=1,2,3$, and $\hat{\chi}_m$. This, in turn, leads to a fourth-order determinant equation for Ω^2.

For *Class II* vibrations,

$$[\![\chi_m]\!]=0\Rightarrow\hat{\chi}_m^+=\hat{\chi}_m^-\triangleq\hat{\chi}_m\Rightarrow\chi_m=\hat{\chi}_mP_\tau^m(x)\Rightarrow\langle\chi_{m,\phi}\rangle=0 \quad (89)$$

Hence, $2U_m=m\langle\psi_m\rangle+\langle\chi_{m,\phi}\rangle=0$ implies that

$$\langle\psi_m\rangle=\sum_{i=1}^3\langle{}_i\hat{w}_m\rangle\hat{C}(\nu,\lambda_i,\Omega^2)P(\sigma_i,m)=0 \quad (90)$$

Further, the two *Class II* conditions $[\![(\Delta w_m)_{,\phi}]\!]=W_m=0$ imply that

$$\sum_{i=1}^{3} \langle {}_i\hat{w}_m \rangle \lambda_i S(\sigma_i, m) = 0 \quad \text{and} \quad \sum_{i=1}^{3} \langle {}_i\hat{w}_m \rangle P(\sigma_i, m) = 0 \quad (91)$$

Thus, $\langle {}_i\hat{w}_m \rangle = 0$, $i=1,2,3$, so that ${}_i\hat{w}_m^+ = -{}_i\hat{w}_m^- \triangleq \hat{w}_m^i$.

From Eqs. $(62)_{2,4}$,

$$V_m = \frac{1}{2}\langle \psi_{m,\phi} \rangle + \frac{1}{2} m \langle \chi_m \rangle = \sum_{i=1}^{3} \hat{w}_m^i \hat{C}(\nu, \lambda_i, \Omega^2) S(\sigma_i, m) + m \hat{\chi}_m P(\tau, m) \quad (92)$$

and

$$\Lambda_m = \frac{1}{2}\langle w_{m,\phi} \rangle = \sum_{i=1}^{3} \hat{w}_m^i S(\sigma_i, m) \quad (93)$$

Finally, $[\![w_m]\!] = [\![u_m]\!] = m[\![\psi_m]\!] + [\![\chi_{m,\phi}]\!] = 0$ implies that

$$\sum_{i=1}^{3} \hat{w}_m^i P(\sigma_i, m) = 0 \quad \text{and} \quad m \sum_{i=1}^{3} \hat{w}_m^i \hat{C}(\nu, \lambda_i, \Omega^2) P(\sigma_i, m) + \hat{\chi}_m S(\tau, m) = 0 \quad (94)$$

If Eqs. (92) and (93) are substituted into Eqs. (81) and (83) then, together with Eq. (94), we obtain four homogeneous linear algebraic equation for the four constants $\hat{w}_m^i$, $i=1,2,3$ and $\hat{\chi}_m$. (These are different constants from Class I.) This, in turn, leads to a fourth-order determinant equation for Ω^2.

We have reduced our problem to the solution of two sets of four linear homogeneous algebraic equations. Using the mode shapes associated with these two eigenvalue problems, we may synthesize the response of the shell-beam system to any periodic forcing function with a spatial dependence given by Eq. (65). Such an analysis soon leads to heavy numerical computations. However, because our final set of equations contains several small parameters, perturbation methods are available. Such methods, discussed in Part II of this paper, not only allow us to proceed further analytically, but, more importantly, *they yield insight into the influence of these dimensionless quantities*.

11 A Special Case: The Natural Frequencies of a Circular Elastic Beam

In the absence of the shell, the equations for the free vibrations of an elastic circular beam of rectangular cross section follow from Eqs. (80)–(83) as

$$-k(B^2/12R^2)m(m^2-1)^2 W_m + g\Omega^2(U_m + mW_m) = 0 \quad (95)$$

$$-k(H^2/12R^2)(m^2-1)[(m^2-1)V_m + \Lambda_m] + g\Omega^2 V_m = 0 \quad (96)$$

$$k[(mU_m - W_m)] \underline{- k(B^2/12R^2)(m^2-1)^2 W_m} + g\Omega^2 W_m = 0 \quad (97)$$

and

$$12(GJ/E)(B/H)m^2\Lambda_m + (m^2-1)V_m + \Lambda_m = 0 \quad (98)$$

These equations display three distinct types of natural frequencies.

11.1 Extensional Vibrations: $\Omega^2 = O(1)$. Equations (95) and (97) govern Class I free vibrations. If $\Omega^2 \neq 0$ and terms of relative order B^2/R^2 are neglected, they have nontrivial solutions provided

$$\begin{vmatrix} 1 & m \\ km & -k + g\Omega^2 \end{vmatrix} = 0 \quad (99)$$

Thus, by Eq. (79)

$$\Omega^2 = (k/g)(m^2+1) = (E/E_S)(\rho_S/\rho)(m^2+1) \quad (100)$$

which, except for the manner of nondimensionalizing, agrees with Love's equation (24) on p. 454 of Ref. [2].

Table 1 Lowest dimensionless natural frequencies for beam-shell torsion

g	$(1+\nu)\Omega^2$
0.001	4.9967
0.01	4.9673
0.1	4.6898
1	3.2465
10	2.1793
100	2.0187
1000	2.0019

11.2 Flexural Vibrations: $\Omega^2 = O(B^2/R^2)$. Equations (95) and (97) also govern planar flexural vibrations and have nontrivial solutions provided

$$\begin{vmatrix} g\Omega^2 & gm\Omega^2 - mk(B^2/12R^2)(m^2-1)^2 \\ km & g\Omega^2 - k \end{vmatrix} = 0 \quad (101)$$

whereas Eqs. (96) and (98) govern the out-of-plane flexural vibrations and have nontrivial solutions provided

$$\begin{vmatrix} -(m^2-1)^2 + (12R^2/H^2)(g/k)\Omega^2 & -(m^2-1) \\ m^2-1 & 12(BGJ/EH)m^2+1 \end{vmatrix} = 0 \quad (102)$$

If we neglect terms of relative order B^2/R^2, Eqs. (101) and (102) yield, respectively

$$\Omega^2 = (k/g)(H^2/12R^2)m^2(m^2-1)^2\left\{\frac{1}{m^2+1}, \frac{1}{m^2+(EH/12BGJ)}\right\} \quad (103)$$

These equations for Ω^2 are of the same form as Eqs. (21) and (22) for a beam of circular cross section on pp. 452 and 453 of Ref. [2]. The flexural frequencies are of the order H/R times the extensional frequencies, as to be expected.

12 Another Special Case: Pure Torsional Vibration ($m=0$)

This is a Class I free vibration in which $w_0 = \psi_0 = V_0 = W_o = \Lambda_0 = 0$, U_0 is an arbitrary constant and $\chi_0 \neq 0$.The only nontrivial equation is Eq. (80), which reduces to

$$-[1+(1+\nu)\Omega^2][\![\chi_0]\!] + g(1+\nu)\Omega^2 U_0 = 0 \quad (104)$$

From Eq. (86), $U_0 = 1/2\langle \chi_{0,\phi} \rangle = 1/2[\![\hat{\chi}_0]\!]S(\tau,0)$ and from Eq. (87), $[\![\chi_0]\!] = 2\hat{\chi}_0 P(\tau,0)$. Hence, the natural frequencies satisfy

$$-[1+(1+\nu)\Omega^2]P(\tau,0) + \tfrac{1}{2}\pi g(1+\nu)\Omega^2 S(\tau,0) = 0 \quad (105)$$

or by Eqs. (72) and (73)

$$1+(1+\nu)\Omega^2 = g(1+\nu)\Omega^2 \frac{\Gamma^2\left(1+\frac{1}{2}\tau\right)}{\Gamma^2\left(\frac{1}{2}+\frac{1}{2}\tau\right)} \tan\left(\frac{1}{2}\tau\pi\right) \quad (106)$$

where from Eq. (75)

$$\tau = -\tfrac{1}{2} + \sqrt{\tfrac{9}{4} + 2(1+\nu)\Omega^2} = 1 + (2/3)(1+\nu)\Omega^2 + O(\Omega^4) \quad (107)$$

Table 1 lists values of $(1+\nu)\Omega^2$ corresponding to various values of $g \triangleq (\rho/\rho_S)(BH/hR)$ for the *lowest natural frequency* of the beam shell, as determined from Eq. (106). As $g \to \infty$, the beam becomes relatively massive and the natural frequency is that of a closed spherical shell in which the equator is a nodal line. This corresponds to the case $n=2$ discussed by Niordson on p. 318 of Ref. [1], except that in this limiting mode, the upper and lower hemispheres move in the *same* direction, whereas if the beam is motionless (or absent), there is another $n=2$ mode in which the upper and lower hemispheres move in *opposite* directions. As g decreases, the equator of the shell begins to move in the opposite direction from the upper and lower parts of the shell so that in the limit as $g \to 0$, the natural frequency is again that of a closed spherical shell, but now moving in an $n=3$ mode, as discussed by Niordson on the same page as above.

13 Conclusion

Thanks to the decoupling of the equations governing the natural frequencies of vibration of an elastic spherical shell fixed to an elastic equatorial beam, we need only find the nontrivial solutions of two uncoupled sets of four homogeneous linear algebraic equations. These contain relative errors no greater than those inherent in the classical equations in linear beam and shell theory. However, because the ratios of the beam and shell thicknesses to the midsurface radius of the shell are small, perturbation methods may be used to greatly simplify the equations. These are the focus of Part II of this paper.

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